

NIJENHUIS OPERATORS ON n -LIE ALGEBRAS

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ABSTRACT. In this paper, we study $(n - 1)$ -order deformations of an n -Lie algebra and introduce the notion of a Nijenhuis operator on an n -Lie algebra, which could give rise to trivial deformations. We prove that a polynomial of a Nijenhuis operator is still a Nijenhuis operator. Finally, we give various constructions of Nijenhuis operators and some examples.

1. INTRODUCTION

The notion of a Filippov algebra, or an n -Lie algebra was introduced in [15]. It is the algebraic structure corresponding to Nambu mechanics [16, 22, 25]. n -Lie algebras, or more generally, n -Leibniz algebras, are widely studied [4, 6, 7, 9, 11, 12, 20, 24, 26]. In particular, 3-Lie algebras were studied from several aspects recently [2, 3, 5, 14, 28] due to applications in the Bagger-Lambert-Gustavsson theory of multiple M2-branes [1, 8, 18, 19, 23, 27]. See the review article [10] for more details.

The aim of this paper is to study $(n - 1)$ -order deformations of an n -Lie algebra. In particular, we pay special attention to a trivial deformation, which could give rise to an operator that satisfies some conditions. We call such an operator a Nijenhuis operator on an n -Lie algebra. It is believed that one can learn more about a mathematical object by studying its deformations [17]. Furthermore, Nijenhuis operators on Lie algebras play an important role in the study of integrability of nonlinear evolution equations [13]. Deformations of n -Lie algebras have been studied from several aspects. See [10, 11, 14, 26, 28] for more details. In particular, a notion of a Nijenhuis operator on a 3-Lie algebra was introduced in [28] in the study of the 1-order deformations of a 3-Lie algebra. But there are some quite strong conditions in this definition of a Nijenhuis operator. In the case of Lie algebras, one could obtain fruitful results by considering one-parameter infinitesimal deformations, i.e. 1-order deformations. However, for n -Lie algebras, we believe that one should consider $(n - 1)$ -order deformations to obtain similar results. In [14], for 3-Lie algebras, the author has already considered 2-order deformations. But Nijenhuis operators were not studied there. Our Nijenhuis operators are obtained by considering an $(n - 1)$ -order trivial deformation of an n -Lie algebra. On the other hand, our Nijenhuis operators on 3-Lie algebras match up very well with some other existing interesting operators, such as Rota-Baxter operators [5] and O -operators [2] on 3-Lie algebras.

The paper is organized as follows. In Section 2, we recall some facts on n -Lie algebras, their representations and associated cohomologies. In Section 3, we consider $(n - 1)$ -order deformations of an n -Lie algebra. We give the notion of a Nijenhuis operator on an n -Lie algebra, and show that it could give rise to a trivial deformation (Theorem 3.7). We show that a polynomial of a Nijenhuis operator is still a Nijenhuis operator (Theorem 3.12). Furthermore, our Nijenhuis

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operators match up with O -operators on n -Lie algebras (Proposition 3.15). In Section 4, according to constructions of n -Lie algebras, we give various constructions of Nijenhuis operators. We also give examples of Nijenhuis operators on some 4-dimensional 3-Lie algebras as a guide for a further development.

2. PRELIMINARIES

Definition 2.1. An n -Lie algebra \mathfrak{g} is a vector space together with an n -multilinear skew-symmetric bracket $[\cdot, \dots, \cdot] : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all $x_i, y_i \in \mathfrak{g}$, the following Filippov identity is satisfied:

$$(1) \quad [x_1, x_2, \dots, x_{n-1}, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n [y_1, y_2, \dots, [x_1, x_2, \dots, x_{n-1}, y_i], \dots, y_n].$$

For $x_1, x_2, \dots, x_{n-1} \in \mathfrak{g}$, define $\text{ad} : \wedge^{n-1} \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad}_{x_1, x_2, \dots, x_{n-1}} y = [x_1, x_2, \dots, x_{n-1}, y], \quad \forall y \in \mathfrak{g}.$$

Then Eq. (1) is equivalent to that $\text{ad}_{x_1, x_2, \dots, x_{n-1}}$ is a derivation, i.e.

$$\text{ad}_X [y_1, y_2, \dots, y_n] = \sum_{i=1}^n [y_1, y_2, \dots, \text{ad}_X y_i, \dots, y_n], \quad \forall X = (x_1, x_2, \dots, x_{n-1}) \in \wedge^{n-1} \mathfrak{g}.$$

Elements in $\wedge^{n-1} \mathfrak{g}$ are called **fundamental objects** of the n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$. In the sequel, we will denote $\text{ad}_X y$ simply by $X \circ y$.

Define a bilinear operation on the set of fundamental objects $\circ : (\wedge^{n-1} \mathfrak{g}) \otimes (\wedge^{n-1} \mathfrak{g}) \longrightarrow \wedge^{n-1} \mathfrak{g}$ by

$$(2) \quad X \circ Y = \sum_{i=1}^{n-1} (y_1, \dots, y_{i-1}, X \circ y_i, y_{i+1}, \dots, y_{n-1}),$$

for all $X = (x_1, x_2, \dots, x_{n-1})$ and $Y = (y_1, y_2, \dots, y_{n-1})$. In [9], the authors proved that $(\wedge^{n-1} \mathfrak{g}, \circ)$ is a Leibniz algebra, i.e. the following equality holds:

$$(3) \quad X \circ (Y \circ Z) = (X \circ Y) \circ Z + Y \circ (X \circ Z), \quad \forall X, Y, Z \in \wedge^{n-1} \mathfrak{g}.$$

Moreover, the Filippov identity (1) is equivalent to

$$(4) \quad X \circ (Y \circ z) - Y \circ (X \circ z) = (X \circ Y) \circ z, \quad \forall X, Y \in \wedge^{n-1} \mathfrak{g}, z \in \mathfrak{g}.$$

Definition 2.2. Let V be a vector space. A representation of an n -Lie algebra \mathfrak{g} on V is a multilinear map $\rho : \wedge^{n-1} \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, such that for all $X, Y \in \wedge^{n-1} \mathfrak{g}$, $x_i, y_i \in \mathfrak{g}$, the following equalities hold:

$$\begin{aligned} [\rho(X), \rho(Y)] &= \rho(X \circ Y), \\ \rho(x_1, x_2, \dots, x_{n-2}, [y_1, y_2, \dots, y_n]) &= \sum_{i=1}^n (-1)^{n-i} \rho(y_1, \dots, \hat{y}_i, \dots, y_n) \rho(x_1, \dots, x_{n-2}, y_i), \end{aligned}$$

where \hat{y}_i means that y_i is omitted.

We denote a representation by $(V; \rho)$.

Given a representation $(V; \rho)$, there is a semidirect product n -Lie algebra structure on $\mathfrak{g} \oplus V$ given by

$$[x_1 + v_1, \dots, x_n + v_n] = [x_1, \dots, x_n] + \sum_{i=1}^n (-1)^{n-i} \rho(x_1, \dots, \hat{x}_i, \dots, x_n)(v_i), \quad \forall x_i \in \mathfrak{g}, v_i \in V.$$

We denote this semidirect n -Lie algebra simply by $\mathfrak{g} \ltimes_{\rho} V$. In particular, when $n = 2$, i.e. for a Lie algebra, we obtain the usual notion of a semidirect product Lie algebra.

A p -cochain on \mathfrak{g} with the coefficients in a representation $(V; \rho)$ is a multilinear map $\alpha^p : \wedge^{n-1} \mathfrak{g} \otimes \binom{p-1}{\cdot} \otimes \wedge^{n-1} \mathfrak{g} \wedge \mathfrak{g} \rightarrow V$. Denote the space of p -cochains by $C^p(\mathfrak{g}; V)$. The coboundary operator $\delta : C^p(\mathfrak{g}; V) \rightarrow C^{p+1}(\mathfrak{g}; V)$ is given by

$$\begin{aligned} & (\delta \alpha^p)(X_1, \dots, X_p, z) \\ &= \sum_{1 \leq i < k} (-1)^i \alpha^p(X_1, \dots, \hat{X}_i, \dots, X_{k-1}, X_i \circ X_k, X_{k+1}, \dots, X_p, z) \\ &+ \sum_{i=1}^p (-1)^i \alpha^p(X_1, \dots, \hat{X}_i, \dots, \mathfrak{X}_p, [X_i, z]) \\ &+ \sum_{i=1}^p (-1)^{i+1} \rho(X_i) \alpha^p(X_1, \dots, \hat{X}_i, \dots, X_p, z) \\ &+ \sum_{i=1}^{n-1} (-1)^{n+p-i+1} \rho(x_p^1, x_p^2, \dots, \hat{x}_p^i, \dots, x_p^{n-1}, z) \alpha^p(X_1, \dots, X_{p-1}, x_p^i), \end{aligned}$$

for all $X_i = (x_i^1, x_i^2, \dots, x_i^{n-1}) \in \wedge^{n-1} \mathfrak{g}$ and $z \in \mathfrak{g}$.

3. NIJENHUIS OPERATORS ON n -LIE ALGEBRAS

In this section, first we study $(n-1)$ -order deformations of an n -Lie algebra, and introduce the notion of a Nijenhuis operator on an n -Lie algebra, which could generate a trivial deformation. Then we show that a polynomial of a Nijenhuis operator is still a Nijenhuis operator. Finally, we give the relation between \mathcal{O} -operators and Nijenhuis operators.

3.1. $(n-1)$ -order deformations of an n -Lie algebra. Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra. Let $\omega_i : \otimes^n \mathfrak{g} \rightarrow \mathfrak{g}$, $1 \leq i \leq n-1$ be skew-symmetric multilinear maps. Consider a λ -parametrized family of linear operations:

$$(5) \quad [x_1, x_2, \dots, x_{n-1}, x_n]_{\lambda} = [x_1, x_2, \dots, x_{n-1}, x_n] + \sum_{i=1}^{n-1} \lambda^i \omega_i(x_1, x_2, \dots, x_{n-1}, x_n).$$

Here $\lambda \in \mathbb{F}$, where \mathbb{F} is the base field. If all $[\cdot, \dots, \cdot]_{\lambda}$ are n -Lie algebra structures, we say that $\omega_1, \dots, \omega_{n-1}$ generate an $(n-1)$ -order 1-parameter deformation of the n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$.

Proposition 3.1. *With the above notations, $\omega_1, \dots, \omega_{n-1}$ generate an $(n-1)$ -order 1-parameter deformation of the n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$ if and only if the following conditions are satisfied:*

$$(6) \quad \delta\omega_1 = 0;$$

$$(7) \quad \delta\omega_l + \frac{1}{2} \sum_{i=1}^{l-1} [\omega_i, \omega_{l-i}] = 0, \quad 2 \leq l \leq n-1;$$

$$(8) \quad \frac{1}{2} \sum_{i=l-n+1}^{n-1} [\omega_i, \omega_{l-i}] = 0, \quad n \leq l \leq 2n-2.$$

Here $[\omega_i, \omega_j]$ is given by

$$(9) \quad \begin{aligned} [\omega_i, \omega_j](X, Y, z) &= \omega_i(X, \omega_j(Y, z)) - \omega_i(Y, \omega_j(X, z)) + \omega_j(X, \omega_i(Y, z)) - \omega_j(Y, \omega_i(X, z)) \\ &\quad - \omega_i(\omega_j(X, \cdot) \circ Y, z) - \omega_j(\omega_i(X, \cdot) \circ Y, z), \quad \forall X, Y \in \wedge^{n-1} \mathfrak{g}, z \in \mathfrak{g}, \end{aligned}$$

where $\omega_j(X, \cdot) \circ Y \in \wedge^{n-1} \mathfrak{g}$ is given by

$$\omega_j(X, \cdot) \circ Y = \sum_{k=1}^{n-1} y_1 \wedge \dots \wedge \omega_j(X, y_k) \wedge \dots \wedge y_{n-1}, \quad \forall Y = (y_1, \dots, y_{n-1}).$$

Proof. All $[\cdot, \dots, \cdot]_\lambda$ are n -Lie algebra structures if and only if

$$X \circ_\lambda (Y \circ_\lambda z) - (X \circ_\lambda Y) \circ_\lambda z - Y \circ_\lambda (X \circ_\lambda z) = 0, \quad \forall X, Y \in \wedge^{n-1} \mathfrak{g}, z \in \mathfrak{g}.$$

First we have

$$Y \circ_\lambda z = Y \circ z + \sum_{i=1}^{n-1} \lambda^i \omega_i(Y, z); \quad X \circ_\lambda Y = X \circ Y + \sum_{i=1}^{n-1} \lambda^i \omega_i(X, \cdot) \circ Y.$$

Then by direct computations, we have

$$\begin{aligned} X \circ_\lambda (Y \circ_\lambda z) &= X \circ (Y \circ z) + \sum_{i=1}^{n-1} \lambda^i (\omega_i(X, Y \circ z) + X \circ \omega_i(Y, z)) \\ &\quad + \sum_{i,j=1}^{n-1} \lambda^{i+j} \omega_i(X, \omega_j(Y, z)); \\ Y \circ_\lambda (X \circ_\lambda z) &= Y \circ (X \circ z) + \sum_{i=1}^{n-1} \lambda^i (\omega_i(Y, X \circ z) + Y \circ \omega_i(X, z)) \\ &\quad + \sum_{i,j=1}^{n-1} \lambda^{i+j} \omega_i(Y, \omega_j(X, z)); \\ (X \circ_\lambda Y) \circ_\lambda z &= (X \circ Y) \circ z + \sum_{i=1}^{n-1} \lambda^i (\omega_i(X \circ Y, z) + (\omega_i(X, \cdot) \circ Y) \circ z) \\ &\quad + \sum_{i,j=1}^{n-1} \lambda^{i+j} \omega_i(\omega_j(X, \cdot) \circ Y) \circ z. \end{aligned}$$

Comparing the coefficients of λ^l , $1 \leq l \leq 2n - 2$, we obtain conditions (6)-(8) respectively. ■

Remark 3.2. *The bracket given by Eq. (9) is just the Nijenhuis-Richardson bracket associated to an n -Lie algebra. See [24] for more details.*

Definition 3.3. *A deformation is said to be **trivial** if there exists a linear map $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for all λ , $T_\lambda = \text{id} + \lambda N$ satisfies*

$$(10) \quad T_\lambda[x_1, x_2, \dots, x_n]_\lambda = [T_\lambda x_1, T_\lambda x_2, \dots, T_\lambda x_n], \quad \forall x_1, \dots, x_n \in \mathfrak{g}.$$

The left hand side of Eq. (10) equals to

$$\begin{aligned} & [x_1, x_2, \dots, x_n] + \lambda(\omega_1(x_1, x_2, \dots, x_n) + N[x_1, x_2, \dots, x_n]) \\ & + \sum_{j=1}^{n-2} \lambda^{j+1}(\omega_{j+1}(x_1, x_2, \dots, x_n) + N\omega_j(x_1, x_2, \dots, x_n)) + \lambda^n N\omega_{n-1}(x_1, x_2, \dots, x_n). \end{aligned}$$

The right hand side of Eq. (10) equals to

$$\begin{aligned} & [x_1, x_2, \dots, x_n] + \lambda \sum_{i=1}^n [x_1, \dots, Nx_i, \dots, x_n] + \lambda^2 \sum_{i < j} [x_1, \dots, Nx_i, \dots, Nx_j, \dots, x_n] \\ & + \lambda^3 \sum_{i < j < k} [x_1, \dots, Nx_i, \dots, Nx_j, \dots, Nx_k, \dots, x_n] + \dots + \lambda^n [Nx_1, Nx_2, \dots, Nx_n]. \end{aligned}$$

Therefore, by Eq. (10), we have

$$(11) \quad \omega_1(x_1, x_2, \dots, x_n) + N[x_1, x_2, \dots, x_n] = \sum_{i=1}^n [x_1, \dots, Nx_i, \dots, x_n],$$

$$(12) \quad N\omega_{n-1}(x_1, x_2, \dots, x_n) = [Nx_1, Nx_2, \dots, Nx_n],$$

and

$$(13) \quad \omega_l(x_1, x_2, \dots, x_n) + N\omega_{l-1}(x_1, x_2, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_l} [\dots, Nx_{i_1}, \dots, Nx_{i_k}, \dots, Nx_{i_l}, \dots],$$

for all $2 \leq l \leq n - 1$.

Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra, and $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ a linear map. Define an n -ary bracket $[\cdot, \dots, \cdot]_N^1 : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$(14) \quad [x_1, x_2, \dots, x_n]_N^1 = \sum_{i=1}^n [x_1, \dots, Nx_i, \dots, x_n] - N[x_1, x_2, \dots, x_n].$$

Then we define n -ary brackets $[\cdot, \dots, \cdot]_N^j : \wedge^n \mathfrak{g} \longrightarrow \mathfrak{g}$, ($2 \leq j \leq n - 1$) via induction by

$$(15) \quad [x_1, x_2, \dots, x_n]_N^j = \sum_{i_1 < i_2 < \dots < i_j} [\dots, Nx_{i_1}, \dots, Nx_{i_j}, \dots] - N[x_1, x_2, \dots, x_n]_N^{j-1}.$$

In particular, we have

$$(16) \quad [x_1, x_2, \dots, x_n]_N^{n-1} = \sum_{i_1 < i_2 < \dots < i_{n-1}} [\dots, Nx_{i_1}, \dots, Nx_{i_{n-1}}, \dots] - N[x_1, x_2, \dots, x_n]_N^{n-2}.$$

Definition 3.4. Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra. A linear map $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a **Nijenhuis operator** if

$$(17) \quad [Nx_1, Nx_2, \dots, Nx_n] = N([x_1, x_2, \dots, x_n]_N^{n-1}), \quad \forall x_1, \dots, x_n \in \mathfrak{g}.$$

Note that when $n = 2$, i.e. for a Lie algebra, we obtain the usual notion of a Nijenhuis operator on a Lie algebra. More precisely, a linear transformation $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ if the following equality holds:

$$[Nx, Ny] = N([Nx, y] + [x, Ny] - N[x, y]), \quad \forall x, y \in \mathfrak{g}.$$

Remark 3.5. In [10, 11, 12], the authors considered deformations of the form $[\cdot, \dots, \cdot] + \lambda\omega(\cdot, \dots, \cdot)$. In [14], for 3-Lie algebras, the author has considered deformations of the form of Eq. (5). But Nijenhuis operators were not considered there. On the other hand, in [28], the author has introduced a notion of a Nijenhuis operator on a 3-Lie algebra in the study of 1-order trivial deformations. In that definition, there is a quite strong condition $[Nx_1, Nx_2, N_3] = 0$, whereas the above definition for $n = 3$ is $[Nx_1, Nx_2, N_3] = N([x_1, x_2, x_3]_N^2)$. So obviously, the above definition is different with these studies.

By Eqs. (15) and (17), we have

Proposition 3.6. Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra. Then N is a Nijenhuis operator on \mathfrak{g} if and only if N satisfies the following equality

$$(18) \quad \sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^p [x_{\sigma(1)}, \dots, x_{\sigma(p)}, Nx_{\sigma(p+1)}, \dots, Nx_{\sigma(n)}] = 0, \quad \forall x_i \in \mathfrak{g},$$

where the summation is taken over all $(p, n-p)$ -unshuffles, i.e. $\sigma(1) < \dots < \sigma(p)$, $\sigma(p+1) < \dots < \sigma(n)$.

By Eqs. (11)-(13), it is straightforward to deduce that a trivial deformation gives rise to a Nijenhuis operator. The following theorem shows that the converse is also true.

Theorem 3.7. Let N be a Nijenhuis operator on an n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$. Then a deformation can be obtained by putting

$$(19) \quad \omega_i(x_1, x_2, \dots, x_n) = [x_1, x_2, \dots, x_n]_N^i, \quad 1 \leq i \leq n-1.$$

Moreover, this deformation is trivial.

One way to prove this theorem directly is to verify that conditions in Proposition 3.1 are satisfied. Instead we apply a different method to prove this theorem to avoid complicated and lengthy computations. The following general fact is an important ingredient in the proof.

Lemma 3.8. Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra and \mathfrak{h} a vector space with an n -ary bracket $[\cdot, \dots, \cdot]'$. If there exists an isomorphism between vector spaces, say $f : \mathfrak{h} \longrightarrow \mathfrak{g}$, such that

$$f[x_1, x_2, \dots, x_n]' = [f(x_1), f(x_2), \dots, f(x_n)], \quad \forall x_i \in \mathfrak{h},$$

then $(\mathfrak{h}, [\cdot, \dots, \cdot]')$ is an n -Lie algebra.

Proof. It follows from straightforward computations. ■

The proof of Theorem 3.7: It is obvious that for a Nijenhuis operator N , the maps $\omega_1, \dots, \omega_{n-1}$ given by Eq. (19) satisfy Eq. (17). Therefore, for any λ , T_λ satisfies

$$T_\lambda[x_1, x_2, \dots, x_n]_\lambda = [T(x_1), T(x_2), \dots, T(x_n)], \quad \forall x_1, \dots, x_n \in \mathfrak{g}.$$

For λ sufficiently small, we see that T_λ is an isomorphism between vector spaces. By Lemma 3.8, we deduce that $(\mathfrak{g}, [\cdot, \dots, \cdot]_\lambda)$ is an n -Lie algebra, for λ sufficiently small. Thus, $\omega_1, \dots, \omega_{n-1}$ given by Eq. (19) satisfy the conditions (6)-(8) in Proposition 3.1. Therefore, $(\mathfrak{g}, [\cdot, \dots, \cdot]_\lambda)$ is an n -Lie algebra for all λ , which means that $\omega_1, \dots, \omega_{n-1}$ given by Eq. (19) generate a deformation. It is obvious that this deformation is trivial. ■

Corollary 3.9. *Let N be a Nijenhuis operator on an n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$. Then $(\mathfrak{g}, [\cdot, \dots, \cdot]_N^{n-1})$ is an n -Lie algebra, and N is a homomorphism from $(\mathfrak{g}, [\cdot, \dots, \cdot]_N^{n-1})$ to $(\mathfrak{g}, [\cdot, \dots, \cdot])$.*

At the end of this subsection, as an example, we study Nijenhuis operators on 3-dimensional complex 3-Lie algebras. Recall that a linear map N acting on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ is a Nijenhuis operator if

$$(20) \quad [Nx, Ny, Nz] = N([x, y, z]_N^2),$$

where the 3-ary bracket $[\cdot, \cdot, \cdot]_N^2$ is defined by

$$(21) \quad [x, y, z]_N^2 = [Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] - N[x, y, z]_N^1,$$

where the 3-ary bracket $[\cdot, \cdot, \cdot]_N^1$ is defined by

$$(22) \quad [x, y, z]_N^1 = [Nx, y, z] + [x, Ny, z] + [x, y, Nz] - N[x, y, z].$$

It is obvious that any linear transformation on an abelian 3-Lie algebra is a Nijenhuis operator. On the other hand, it is known that up to isomorphism, there is only one 3-dimensional non-abelian complex 3-Lie algebra given by

$$(23) \quad [e_1, e_2, e_3] = e_1,$$

where $\{e_1, e_2, e_3\}$ is a basis of \mathfrak{g} .

Theorem 3.10. *Let $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ be the 3-dimensional complex 3-Lie algebra given above. Then any linear transformation N on \mathfrak{g} is a Nijenhuis operator.*

Proof. Assume $Ne_i = N_i^j e_j$. Then we have

$$\begin{aligned} [e_1, e_2, e_3]_N^1 &= N_1^1 e_1 + N_2^2 e_1 + N_3^3 e_1 - N_1^j e_j, \\ N[e_1, e_2, e_3]_N^1 &= N_1^1 N_1^j e_j + N_2^2 N_1^j e_j + N_3^3 N_1^j e_j - N_1^j N_j^k e_k. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} [e_1, e_2, e_3]_N^2 &= (N_2^2 N_3^3 - N_2^3 N_3^2) e_1 + (N_1^3 N_3^2 - N_3^3 N_1^2) e_2 + (N_1^2 N_2^3 - N_2^2 N_1^3) e_3, \\ N[e_1, e_2, e_3]_N^2 &= (N_1^1 (N_2^2 N_3^3 - N_2^3 N_3^2) + N_2^2 (N_1^3 N_3^2 - N_3^3 N_1^2) + N_3^3 (N_1^2 N_2^3 - N_2^2 N_1^3)) e_1 \\ &\quad + (N_1^2 (N_2^2 N_3^3 - N_2^3 N_3^2) + N_2^2 (N_1^3 N_3^2 - N_3^3 N_1^2) + N_3^2 (N_1^2 N_2^3 - N_2^2 N_1^3)) e_2 \\ &\quad + (N_1^3 (N_2^2 N_3^3 - N_2^3 N_3^2) + N_2^3 (N_1^3 N_3^2 - N_3^3 N_1^2) + N_3^3 (N_1^2 N_2^3 - N_2^2 N_1^3)) e_3 \\ &= (N_1^1 (N_2^2 N_3^3 - N_2^3 N_3^2) + N_2^2 (N_1^3 N_3^2 - N_3^3 N_1^2) + N_3^3 (N_1^2 N_2^3 - N_2^2 N_1^3)) e_1. \end{aligned}$$

However, we have

$$\begin{aligned} & [Ne_1, Ne_2, Ne_3] \\ &= (N_1^1 N_2^2 N_3^3 - N_1^1 N_2^3 N_3^2 + N_2^1 N_1^3 N_3^2 - N_2^1 N_3^3 N_1^2 + N_3^1 N_1^2 N_2^3 - N_3^1 N_2^2 N_1^3) e_1. \end{aligned}$$

Therefore, we have

$$[Ne_1, Ne_2, Ne_3] = N[e_1, e_2, e_3]_N^2.$$

The proof is finished. ■

3.2. Some properties of Nijenhuis operators.

Lemma 3.11. *Let $N : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Nijenhuis operator on an n -Lie algebra \mathfrak{g} . For all $x_1, x_2, \dots, x_n \in \mathfrak{g}$ and arbitrary positive number $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}$ there holds*

$$(24) \quad \sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] = 0,$$

where the summation is taken over all $(p, n-p)$ -unshuffles. If N is invertible, this formula is valid for arbitrary $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}$.

Proof. The proof is lengthy and nontrivial. We put it in Appendix for self-contained. ■

Theorem 3.12. *Let $N : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Nijenhuis operator on an n -Lie algebra \mathfrak{g} . Then for any polynomial $P(z) = \sum_{i=0}^n c_i z^i$, the operator $P(N)$ is also a Nijenhuis operator. Furthermore, if N is invertible, for any $Q(z) = \sum_{i=-m}^n c_i z^i$, $Q(N)$ is also a Nijenhuis operator.*

Proof. For all $x_1, x_2, \dots, x_n \in \mathfrak{g}$, by Eq. (24), we have

$$\begin{aligned} & \sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} P(N)^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, P(N)x_{\sigma(p+1)}, \dots, P(N)x_{\sigma(n)}] \\ &= \sum_{\alpha_i, 1 \leq i \leq n} \prod_{1 \leq i \leq n} c_{\alpha_i} \left(\sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \right) \\ &= 0. \end{aligned}$$

Therefore, $P(N)$ is a Nijenhuis operator. The second statement can be proved similarly. ■

Remark 3.13. *In some sense, the above property “characterize” a Nijenhuis operator, whereas some known operators like derivations and Rota-Baxter operators do not have such a property.*

In the sequel, we give the relation between O -operators and Nijenhuis operators.

Definition 3.14. *Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra and $(V; \rho)$ a representation. A linear map $T : V \rightarrow \mathfrak{g}$ is called an **O -operator** if for all $v_1, v_2, \dots, v_n \in V$,*

$$(25) \quad [Tv_1, Tv_2, \dots, Tv_n] = \sum_{i=1}^n (-1)^{n-i} T(\rho(Tv_1, \dots, \widehat{Tv_i}, \dots, Tv_n)(v_i)).$$

In particular, if we take the adjoint representation, then an \mathcal{O} -operator is exactly a Rota-Baxter operator of weight 0 given in [5].

In the case of Lie algebras, the notion of an \mathcal{O} -operator was introduced by Kupershmidt in [21] in the study of classical Yang-Baxter equation. It is straightforward to deduce that given a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, an \mathcal{O} -operator $T : V \longrightarrow \mathfrak{g}$ on a Lie algebra \mathfrak{g} could give rise to a Nijenhuis operator $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ on the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\rho} V$.

Similarly, we have

Proposition 3.15. *Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra and $(V; \rho)$ a representation. A linear operator $T : V \rightarrow \mathfrak{g}$ is an \mathcal{O} -operator if and only if*

$$\overline{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : \mathfrak{g} \oplus V \longrightarrow \mathfrak{g} \oplus V$$

is a Nijenhuis operator acting on the semidirect product n -Lie algebra $\mathfrak{g} \ltimes_{\rho} V$.

Proof. For all $x_i \in \mathfrak{g}$, $v_i \in V$, $i = 1, 2, \dots, n$, we have

$$[\overline{T}(x_1 + v_1), \dots, \overline{T}(x_n + v_n)] = [Tv_1, \dots, Tv_n].$$

On the other hand, since $\overline{T}^2 = 0$, we have

$$\begin{aligned} & \overline{T}[x_1 + v_1, \dots, x_n + v_n]_{\overline{T}}^{n-1} \\ &= \overline{T} \left(\sum_{i_1 < i_2 < \dots < i_{n-1}} [\dots, \overline{T}(x_{i_1} + v_{i_1}), \dots, \overline{T}(x_{i_{n-1}} + v_{i_{n-1}}), \dots] \right. \\ & \quad \left. - \overline{T}^2[x_1 + v_1, \dots, x_n + v_n]_{\overline{T}}^{n-2} \right) \\ &= \overline{T}([Tv_1, Tv_2, \dots, Tv_{n-1}, x_n] + c.p. + [Tv_1, Tv_2, \dots, Tv_{n-1}, v_n] + c.p.) \\ &= T \left(\sum_{i=1}^n (-1)^{n-i} \rho(Tv_1, \dots, \widehat{Tv_i}, \dots, Tv_n)(v_i) \right), \end{aligned}$$

which implies that \overline{T} is a Nijenhuis operator if and only if Eq. (25) is satisfied. ■

Remark 3.16. *In fact, when $n = 2$, it is exactly the formerly mentioned conclusion for Lie algebras. Thus, from this point of view, our Nijenhuis operator on an n -Lie algebra is a natural generalization of the one on a Lie algebra, whereas the other so-called Nijenhuis operators (for example, the ones in [28]) do not have this property.*

4. CONSTRUCTIONS OF NIJENHUIS OPERATORS

4.1. Constructions of Nijenhuis operators on $(n + 1)$ -Lie algebras from those on n -Lie algebras. In [4], the authors constructed an $(n + 1)$ -Lie algebra \mathfrak{g}_f from an n -Lie algebras \mathfrak{g} using a linear function f . In this subsection, we show that a Nijenhuis operator on \mathfrak{g} is also a Nijenhuis operator on the $(n + 1)$ -Lie algebra \mathfrak{g}_f .

Lemma 4.1. [4] Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -Lie algebra and \mathfrak{g}^* the dual space of \mathfrak{g} . Suppose $f \in \mathfrak{g}^*$ satisfying $f([x_1, \dots, x_n]) = 0$ for all $x_i \in \mathfrak{g}$. Then there is an $(n + 1)$ -Lie algebra structure on \mathfrak{g}

given by

$$(26) \quad \{x_1, \dots, x_{n+1}\} = \sum_{i=1}^{n+1} (-1)^{i-1} f(x_i)[x_1, \dots, \hat{x}_i, \dots, x_n], \quad \forall x_i \in \mathfrak{g}.$$

We denote it by \mathfrak{g}_f .

Theorem 4.2. *Assume that N is a Nijenhuis operator on an n -Lie algebra $(\mathfrak{g}, [\cdot, \dots, \cdot])$. Then N is also a Nijenhuis operator on the $(n+1)$ -Lie algebra $(\mathfrak{g}_f, \{\cdot, \dots, \cdot\})$.*

Proof. First, for $1 \leq i \leq n-1$, we have

$$(27) \quad \{x_1, x_2, \dots, x_{n+1}\}_N^i = \sum_{j=1}^{n+1} (-1)^{i-1} (f(Nx_j)[x_1, x_2, \dots, \hat{N}x_j, \dots, x_{n+1}]_N^{i-1} + f(x_j)[x_1, x_2, \dots, \hat{x}_j, \dots, x_{n+1}]_N^i).$$

This fact can be proved by induction on i . For $i = 1$, we have

$$\begin{aligned} & \{x_1, x_2, \dots, x_{n+1}\}_N^1 \\ &= \sum_{i=1}^{n+1} \{x_1, \dots, Nx_i, \dots, x_{n+1}\} - N\{x_1, x_2, \dots, x_{n+1}\} \\ &= \sum_{i,j,i \neq j} (-1)^{j-1} f(x_j)[x_1, \dots, \hat{x}_j, \dots, Nx_i, \dots, x_{n+1}] \\ & \quad + \sum_{i=1}^{n+1} (-1)^{i-1} f(Nx_i)[x_1, x_2, \dots, \hat{N}x_i, \dots, x_{n+1}] \\ & \quad - \sum_{i=1}^{n+1} (-1)^{i-1} f(x_i)[x_1, x_2, \dots, \hat{x}_i, \dots, x_{n+1}] \\ &= \sum_i (-1)^{i-1} (f(Nx_i)[x_1, x_2, \dots, \hat{N}x_i, \dots, x_{n+1}] + f(x_i)[x_1, x_2, \dots, \hat{x}_i, \dots, x_{n+1}]_N^1). \end{aligned}$$

Now we assume that Eq. (27) holds for arbitrary i . Then for $i+1$, we have

$$\begin{aligned} & \{x_1, x_2, \dots, x_{n+1}\}_N^{i+1} \\ &= \sum_{j_1 < j_2 < \dots < j_{i+1}} \{\dots, Nx_{j_1}, \dots, Nx_{j_k}, \dots, Nx_{j_{i+1}}, \dots\} - N\{x_1, x_2, \dots, x_n\}_N^i \\ &= \sum_{j_1 < j_2 < \dots < j_{i+1}} \sum_{k \neq j_1, \dots, j_{i+1}} (-1)^{k-1} f(x_k)[\dots, Nx_{j_1}, \dots, \hat{x}_k, \dots, Nx_{j_k}, \dots, Nx_{j_{i+1}}, \dots] \\ & \quad + \sum_{j_1 < j_2 < \dots < j_{i+1}} \sum_{k=j_1, \dots, j_{i+1}} (-1)^{k-1} f(Nx_k)[\dots, Nx_{j_1}, \dots, \hat{N}x_k, \dots, Nx_{j_k}, \dots, Nx_{j_{i+1}}, \dots] \\ & \quad - \sum_k (-1)^{k-1} N(f(Nx_k)[x_1, x_2, \dots, \hat{N}x_k, \dots, x_{n+1}]_N^{i-1} + f(x_k)[x_1, x_2, \dots, \hat{x}_k, \dots, x_{n+1}]_N^i) \\ &= \sum_{j=1}^{n+1} (-1)^{j-1} (f(Nx_j)[x_1, x_2, \dots, \hat{N}x_j, \dots, x_{n+1}]_N^i + f(x_j)[x_1, x_2, \dots, \hat{x}_j, \dots, x_{n+1}]_N^{i+1}), \end{aligned}$$

which implies that Eq. (27) holds. In particular, we have

$$\begin{aligned} & \{x_1, x_2, \dots, x_{n+1}\}_N^{n-1} \\ &= \sum_{j=1}^{n+1} (-1)^{j-1} (f(Nx_j)[x_1, x_2, \dots, \hat{N}x_j, \dots, x_{n+1}]_N^{n-2} + f(x_j)[x_1, x_2, \dots, \hat{x}_j, \dots, x_n]_N^{n-1}). \end{aligned}$$

Since N is a Nijenhuis operator on the n -Lie algebra \mathfrak{g} , we have

$$\begin{aligned} & \{x_1, x_2, \dots, x_{n+1}\}_N^n \\ &= \sum_{j_1 < j_2 < \dots < j_n} \{\dots, Nx_{j_1}, \dots, Nx_{j_k}, \dots, Nx_{j_n}, \dots\} - N\{x_1, x_2, \dots, x_{n+1}\}_N^{n-1} \\ &= \sum_{j_1 < j_2 < \dots < j_n} \{\dots, Nx_{j_1}, \dots, Nx_{j_k}, \dots, Nx_{j_n}, \dots\} \\ & \quad - N \sum_{i=1}^{n+1} (-1)^{i-1} (f(Nx_i)[x_1, x_2, \dots, \hat{N}x_i, \dots, x_{n+1}]_N^{n-2} + f(x_i)[x_1, x_2, \dots, \hat{x}_i, \dots, x_n]_N^{n-1}) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} f(Nx_i)[x_1, x_2, \dots, \hat{N}x_i, \dots, x_{n+1}]_N^{n-1}. \end{aligned}$$

Furthermore, we can get

$$\begin{aligned} N\{x_1, x_2, \dots, x_{n+1}\}_N^n &= \sum_{i=1}^{n+1} (-1)^{i-1} f(Nx_i)N[x_1, x_2, \dots, \hat{N}x_i, \dots, x_{n+1}]_N^{n-1} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} f(Nx_i)N[Nx_1, Nx_2, \dots, \hat{N}x_i, \dots, Nx_{n+1}]_N^n \\ &= \{Nx_1, Nx_2, \dots, Nx_{n+1}\}, \end{aligned}$$

which implies that N is a Nijenhuis operator on the $(n+1)$ -Lie algebra $(\mathfrak{g}_f, \{\cdot, \dots, \cdot\})$. ■

4.2. Constructions of Nijenhuis operators on 3-Lie algebras from Nijenhuis operators on commutative associative algebras. In fact, there is a similar study on the Nijenhuis operators on associative algebras. Explicitly, a linear map N acting on an associative algebra (\mathfrak{g}, \cdot) is called a **Nijenhuis operator** if

$$(28) \quad Nx \cdot Ny = N(Nx \cdot y + x \cdot Ny - N(x \cdot y)), \quad \forall x, y \in \mathfrak{g}.$$

Lemma 4.3. ([5]) *Let (\mathfrak{g}, \cdot) be a commutative associative algebra. Let $D \in \text{Der}(\mathfrak{g})$ and $f \in \mathfrak{g}^*$ satisfy $f(D(x) \cdot y) = f(x \cdot D(y))$. Then $(\mathfrak{g}, \llbracket \cdot, \cdot, \cdot \rrbracket)$ is a 3-Lie algebra, where the bracket is given by*

$$(29) \quad \llbracket x, y, z \rrbracket \triangleq \begin{vmatrix} f(x) & f(y) & f(z) \\ D(x) & D(y) & D(z) \\ x & y & z \end{vmatrix} \triangleq f(x)(D(y) \cdot z - D(z) \cdot y) + c.p..$$

Proposition 4.4. *With the same assumptions as Lemma 4.3. Let N be a Nijenhuis operator on (\mathfrak{g}, \cdot) satisfying $DN = ND$. Then N is a Nijenhuis operator on the 3-Lie algebra $(\mathfrak{g}, \llbracket \cdot, \cdot, \cdot \rrbracket)$, where the bracket is given by Eq. (29).*

Proof. For all $x, y \in \mathfrak{g}$, define $[x, y]_D = D(x) \cdot y - D(y) \cdot x$. By direct calculations, we can verify that $(\mathfrak{g}, [\cdot, \cdot]_D)$ is a Lie algebra. Furthermore, assume that N is a Nijenhuis operator on (\mathfrak{g}, \cdot) satisfying $DN = ND$. Then we have

$$\begin{aligned}
 [Nx, Ny]_D &= DNx \cdot Ny - Nx \cdot D Ny \\
 &= NDx \cdot Ny - Nx \cdot NDy \\
 &= N(Dx \cdot Ny + NDx \cdot y - N(Dx \cdot y)) - N(Nx \cdot Dy + x \cdot NDy - N(x \cdot Dy)) \\
 &= N(DNx \cdot y - Nx \cdot Dy + Dx \cdot Ny - x \cdot D Ny - N(Dx \cdot y - x \cdot Dy)) \\
 &= N([Nx, y]_D + [x, Ny]_D - N[x, y]_D),
 \end{aligned}$$

which implies that N is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_D)$. By Theorem 4.2, N is a Nijenhuis operator on the 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$. ■

Let (\mathfrak{g}, \cdot) be a commutative associative algebra. For $x_i, y_i, z_i \in \mathfrak{g}$, $i = 1, 2, 3$, denote by

$$\begin{aligned}
 \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \end{vmatrix} &= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \\
 &= x_1 \cdot (y_2 \cdot z_3 - y_3 \cdot z_2) - x_2 \cdot (y_1 \cdot z_3 - y_3 \cdot z_1) + x_3 \cdot (y_1 \cdot z_2 - y_2 \cdot z_1),
 \end{aligned}$$

where \vec{x}, \vec{y} and \vec{z} denote the column vectors.

Lemma 4.5. *Let N be a Nijenhuis operator on a commutative associative algebra (\mathfrak{g}, \cdot) and $N(\vec{x}), N(\vec{y}), N(\vec{z})$ denote the images of the column vectors. Then we have*

$$\begin{aligned}
 \begin{vmatrix} N(\vec{x}) & N(\vec{y}) & N(\vec{z}) \end{vmatrix} &= N \left(\begin{vmatrix} N(\vec{x}) & N(\vec{y}) & \vec{z} \end{vmatrix} + c.p. \right) - N^2 \left(\begin{vmatrix} N(\vec{x}) & \vec{y} & \vec{z} \end{vmatrix} + c.p. \right) \\
 &\quad + N^3 \left(\begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \end{vmatrix} \right).
 \end{aligned}$$

Proof. Since N is a Nijenhuis operator on (\mathfrak{g}, \cdot) , we have

$$\begin{aligned}
 \begin{vmatrix} N(\vec{x}) & N(\vec{y}) & N(\vec{z}) \end{vmatrix} &= \sum_{\sigma \in S_3} \text{sgn}(\sigma) N(x_{\sigma(1)}) N(y_{\sigma(2)}) N(z_{\sigma(3)}) \\
 &= N \left(\sum_{\sigma \in S_3} \text{sgn}(\sigma) N(x_{\sigma(1)}) N(y_{\sigma(2)}) z_{\sigma(3)} + c.p. \right) \\
 &\quad - N^2 \left(\sum_{\sigma \in S_3} \text{sgn}(\sigma) N(x_{\sigma(1)}) y_{\sigma(2)} z_{\sigma(3)} + c.p. \right) \\
 &\quad + N^3 \left(\sum_{\sigma \in S_3} \text{sgn}(\sigma) x_{\sigma(1)} y_{\sigma(2)} z_{\sigma(3)} \right) \\
 &= N \left(\begin{vmatrix} N(\vec{x}) & N(\vec{y}) & \vec{z} \end{vmatrix} + c.p. \right) - N^2 \left(\begin{vmatrix} N(\vec{x}) & \vec{y} & \vec{z} \end{vmatrix} + c.p. \right) \\
 &\quad + N^3 \left(\begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \end{vmatrix} \right).
 \end{aligned}$$

The proof is finished. ■

Lemma 4.6. ([5]) *Let (g, \cdot) be a commutative associative algebra, $D_1, D_2 \in \text{Der}(g)$ satisfying $D_1D_2 = D_2D_1$. Then $(g, \llbracket \cdot, \cdot, \cdot \rrbracket)$ is a 3-Lie algebra, where the bracket is given by*

$$(30) \quad \llbracket x, y, z \rrbracket \triangleq \begin{vmatrix} x & y & z \\ D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \end{vmatrix}, \quad \forall x, y, z \in g.$$

Proposition 4.7. *With the same assumptions as Lemma 4.6. Let N be a Nijenhuis operator on (g, \cdot) satisfying $ND_1 = D_1N$, $ND_2 = D_2N$. Then N is a Nijenhuis operator on the 3-Lie algebra $(g, \llbracket \cdot, \cdot, \cdot \rrbracket)$, where the bracket is given by Eq. (30).*

Proof. Since $ND_1 = D_1N$, $ND_2 = D_2N$, by Lemma 4.5, we have

$$\begin{aligned} \llbracket Nx, Ny, Nz \rrbracket &= \begin{vmatrix} Nx & Ny & Nz \\ D_1(Nx) & D_1(Ny) & D_1(Nz) \\ D_2(Nx) & D_2(Ny) & D_2(Nz) \end{vmatrix} \\ &= \begin{vmatrix} Nx & Ny & Nz \\ ND_1(x) & ND_1(y) & ND_1(z) \\ ND_2(x) & ND_2(y) & ND_2(z) \end{vmatrix} \\ &= N \left(\begin{vmatrix} Nx & Ny & z \\ ND_1(x) & ND_1(y) & D_1(z) \\ ND_2(x) & ND_2(y) & D_2(z) \end{vmatrix} + c.p. \right) \\ &\quad - N^2 \left(\begin{vmatrix} Nx & y & z \\ ND_1(x) & D_1(y) & D_1(z) \\ ND_2(x) & D_2(y) & D_2(z) \end{vmatrix} + c.p. \right) + N^3 \left(\begin{vmatrix} x & y & z \\ D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \end{vmatrix} \right) \\ &= N \left(\begin{vmatrix} Nx & Ny & z \\ D_1(Nx) & D_1(Ny) & D_1(z) \\ D_2(Nx) & D_2(Ny) & D_2(z) \end{vmatrix} + c.p. \right) \\ &\quad - N^2 \left(\begin{vmatrix} Nx & y & z \\ D_1(Nx) & D_1(y) & D_1(z) \\ D_2(Nx) & D_2(y) & D_2(z) \end{vmatrix} + c.p. \right) + N^3 \left(\begin{vmatrix} x & y & z \\ D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \end{vmatrix} \right) \\ &= N(\llbracket Nx, Ny, z \rrbracket + c.p.) - N^2(\llbracket Nx, y, z \rrbracket + c.p.) + N^3(\llbracket x, y, z \rrbracket) \\ &= N(\llbracket x, y, z \rrbracket_N^2). \end{aligned}$$

Thus, N is a Nijenhuis operator on the 3-Lie algebra $(g, \llbracket \cdot, \cdot, \cdot \rrbracket)$. ■

Lemma 4.8. ([5]) *Let (g, \cdot) be a commutative associative algebra. Let $D_i \in \text{Der}(g)$ such that $D_iD_j = D_jD_i$, $i, j = 1, 2, 3$. Then $(g, \llbracket \cdot, \cdot, \cdot \rrbracket)$ is a 3-Lie algebra, where the bracket is given by*

$$(31) \quad \llbracket x, y, z \rrbracket := \begin{vmatrix} D_1(x) & D_1(y) & D_1(z) \\ D_2(x) & D_2(y) & D_2(z) \\ D_3(x) & D_3(y) & D_3(z) \end{vmatrix}, \quad \forall x, y, z \in g.$$

Proposition 4.9. *With the same assumptions as Lemma 4.8. Let N be a Nijenhuis operator on (g, \cdot) satisfying $ND_i = D_iN$, $i, j = 1, 2, 3$. Then N is a Nijenhuis operator on the 3-Lie algebra $(g, \llbracket \cdot, \cdot, \cdot \rrbracket)$, where the bracket is given by Eq. (31).*

Proof. The proof is similar to the proof of Proposition 4.7. We omit details. ■

4.3. Constructions of Nijenhuis operators on 3-Lie algebras from Rota-Baxter operators and derivations and some explicit examples. Recall that a **Rota-Baxter operator** (of weight 0) on a 3-Lie algebra $(g, [\cdot, \cdot, \cdot])$ is a linear map $P : g \longrightarrow g$ such that

$$(32) \quad [Px, Py, Pz] = P([Px, Py, z] + [Px, y, Pz] + [x, Py, Pz]), \quad \forall x, y, z \in g,$$

and a **derivation** on a 3-Lie algebra $(g, [\cdot, \cdot, \cdot])$ is a linear map $D : g \longrightarrow g$ such that

$$(33) \quad D[x, y, z] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz], \quad \forall x, y, z \in g.$$

We denote the sets of Rota-Baxter operators (of weight 0) and derivations of a 3-Lie algebra g by $RB(g)$ and $Der(g)$ respectively. Note that $Der(g)$ is a vector space, where $RB(g)$ is not a vector space (it is only a set!).

The following conclusion is straightforward but very important for constructing Nijenhuis operators.

Lemma 4.10. *Let $(g, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. If a linear transformation N is a derivation, then N is a Nijenhuis operator if and only if N is a Rota-Baxter operator (of weight 0) on g . In particular, if a linear transformation $N \in RB(g) \cap Der(g)$, then N is a Nijenhuis operator.*

Example 4.11. Let g be the 4-dimensional simple complex 3-Lie algebra given by

$$(34) \quad [e_2, e_3, e_4] = e_1, [e_1, e_2, e_4] = e_3, [e_1, e_3, e_4] = e_2, [e_1, e_2, e_3] = e_4,$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis of g . Then by direct computations, we have

$$Der(g) = \left\{ \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ -b & d & 0 & f \\ c & -e & f & 0 \end{bmatrix} \middle| a, b, c, d, e, f \in \mathbb{C} \right\}.$$

Let $N \in Der(g)$. Then we have

$$[e_2, e_3, e_4]_N^1 = 0, [e_1, e_2, e_4]_N^1 = 0, [e_1, e_3, e_4]_N^1 = 0, [e_1, e_2, e_3]_N^1 = 0,$$

In our convention, $Ne_i = N_i^j e_j$. Furthermore, we have

$$\begin{aligned} [e_2, e_3, e_4]_N^2 &= (-d^2 + e^2 - f^2)e_1 + (bd - ce)e_2 + (ad + cf)e_3 + (ae + bf)e_4, \\ [e_1, e_2, e_4]_N^2 &= (ad + cf)e_1 - (ba + ef)e_2 - (a^2 + c^2 - e^2)e_3 - (bc - de)e_4, \\ [e_1, e_3, e_4]_N^2 &= (ce - bd)e_1 + (b^2 - c^2 - f^2)e_2 + (ba + ef)e_3 + (ca + df)e_4, \\ [e_1, e_2, e_3]_N^2 &= -(ae + bf)e_1 + (ac + df)e_2 + (bc - de)e_3 + (b^2 - a^2 - d^2)e_4 \end{aligned}$$

It is straightforward to deduce that $N[x, y, z]_N^2 = [Nx, Ny, Nz]$ for all $x, y, z \in g$. Therefore, N is a Rota-Baxter operator, i.e. $Der(g) \subset RB(g)$. Thus, any $N \in Der(g)$ is a Nijenhuis operator.

By Theorem 3.12, N^2 is also a Nijenhuis operator. However, it is straightforward to deduce that N^2 is neither a derivation nor a Rota-Baxter operator any more.

Example 4.12. Let g be the 4-dimensional complex 3-Lie algebra given by

$$(35) \quad [e_2, e_3, e_4] = e_1, [e_1, e_2, e_4] = e_3, [e_1, e_3, e_4] = e_2,$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis of \mathfrak{g} . Then we have

$$\text{Der}(\mathfrak{g}) = \left\{ \begin{bmatrix} h & a & b & 0 \\ a & h & c & 0 \\ -b & c & h & 0 \\ d & e & f & -h \end{bmatrix} \middle| a, b, c, d, e, f, h \in \mathbb{C} \right\}.$$

In general, a derivation $D \in \text{Der}(\mathfrak{g})$ might not be a Rota-Baxter operator (of weight 0) of \mathfrak{g} any more. On the other hand, denote by $T_1(\mathfrak{g}) \subset \text{Der}(\mathfrak{g})$ and $T_2(\mathfrak{g}) \subset \text{Der}(\mathfrak{g})$ respectively two subspaces of $\text{Der}(\mathfrak{g})$ given by

$$T_1(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & a & b & 0 \\ a & 0 & c & 0 \\ -b & c & 0 & 0 \\ d & e & f & 0 \end{bmatrix} \middle| a, b, c, d, e, f \in \mathbb{C} \right\},$$

and

$$T_2(\mathfrak{g}) = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ b & c & d & -a \end{bmatrix} \middle| a, b, c, d \in \mathbb{C} \right\}.$$

Then $\text{Der}(\mathfrak{g}) = T_1(\mathfrak{g}) + T_2(\mathfrak{g})$. Furthermore, we can deduce that a derivation $D \in \text{Der}(\mathfrak{g})$ is a Rota-Baxter operator (of weight 0) of \mathfrak{g} if and only if $D \in T_1(\mathfrak{g})$, or $D \in T_2(\mathfrak{g})$. Thus, a derivation $D \in \text{Der}(\mathfrak{g})$ is a Nijenhuis operator if and only if $D \in T_1(\mathfrak{g})$, or $D \in T_2(\mathfrak{g})$.

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APPENDIX

The proof of Lemma 3.11: Fix $\alpha_1 = 1, \alpha_2 = 1, \dots, \alpha_{n-1} = 1$ and prove Eq. (24) for arbitrary $\alpha_n > 0$. For $\alpha_n = 1$, the formula is just Eq. (18). Now assume that Eq. (24) holds for $\alpha_n = \beta_n$. By Eq. (18), for $\alpha_n = \beta_n + 1$, we get

$$\begin{aligned} & [Nx_1, Nx_2, \dots, Nx_{n-1}, N^{\beta_n+1}x_n] + \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \\ & [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}}x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}}x_{\sigma(n)}] \\ = & \sum_{p=1}^n \sum_{\sigma} \left(-(-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, Nx_{\sigma(p+1)}, \dots, Nx_{\sigma(n)}] \right. \\ & \left. + (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \right. \\ & \left. [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}}x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}}x_{\sigma(n)}] \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=1}^n \left(- \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, N^{\beta_n} x_n, Nx_{\sigma(p+1)}, \dots, Nx_{\sigma(n)}] \right. \\
&\quad - \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, Nx_{\sigma(p+1)}, \dots, N^{\beta_n+1} x_n] \\
&\quad + \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^{\sum_{j=1}^{p-1} \alpha_{\sigma(j)} + \beta_n + 1} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_n, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad + \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, Nx_{\sigma(p+1)}, \dots, N^{\beta_n+1} x_n] \Big) \\
&= - \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^p [x_{\sigma(1)}, x_{\sigma(2)}, \dots, N^{\beta_n} x_n, Nx_{\sigma(p+1)}, \dots, Nx_{\sigma(n)}] \\
&\quad + \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^{\sum_{j=1}^{p-1} \alpha_{\sigma(j)} + \beta_n + 1} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_n, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&= \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{(p-1)(p-2)}{2} + \sum_{j=1}^{p-1} \sigma(j)} N^p \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p-1)}, Nx_{\sigma(p+1)}, \dots, Nx_{\sigma(n)}, N^{\beta_n} x_n] \\
&\quad + \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^{\sum_{j=1}^{p-1} \alpha_{\sigma(j)} + \beta_n + 1} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_n, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&= N \left([Nx_1, Nx_2, \dots, Nx_{n-1}, N^{\beta_n} x_n] + \sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \right. \\
&\quad \left. [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \right),
\end{aligned}$$

which implies that Eq. (18) holds for $\alpha_1 = 1, \alpha_2 = 1, \dots, \alpha_{n-1} = 1$ and arbitrary positive integer α_n .

Now we assume that Eq. (24) holds for $\alpha_1 = 1, \alpha_2 = 1, \dots, \alpha_{r-1} = 1, \alpha_r = 1$ and arbitrary positive integer $\alpha_{r+1}, \dots, \alpha_{n-1}, \alpha_n$. Then we need to show that Eq. (24) holds for $\alpha_1 = 1, \alpha_2 = 1, \dots, \alpha_{r-1} = 1$ and arbitrary positive integer $\alpha_r, \alpha_{r+1}, \dots, \alpha_{n-1}, \alpha_n$. Let $\alpha_r = \beta_r + 1$, applying Eq. (24) to the element $N^{\beta_r} x_r$ instead of the element x_r and let $y_1 = x_1, \dots, y_r = N^{\beta_r} x_r, \dots, y_n = x_n$. By Eq. (18), we have

$$\begin{aligned}
&\sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad + [Ny_1, Ny_2, \dots, Ny_r, N^{\alpha_{r+1}} y_{r+1}, \dots, N^{\alpha_{n-1}} y_{n-1}, N^{\alpha_n} y_n]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{p=1}^n \left(\sum_{\sigma, \sigma(p+s)=\alpha_r} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \right. \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} y_{\sigma(p+1)}, \dots, N^{\alpha_r+1} x_{\sigma(p+s)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad + \sum_{\sigma, \sigma(s)=\alpha_r} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1, j \neq s}^p \alpha_{\sigma(j)} + \alpha_r + 1} \\
&\quad [x_{\sigma(1)}, \dots, x_{\sigma(s)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad - \sum_{\sigma, \sigma(p+s)=\alpha_r} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)}} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} y_{\sigma(p+1)}, \dots, N^{\alpha_r+1} x_{\sigma(p+s)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad - \sum_{\sigma, \sigma(s)=\alpha_r} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1, j \neq s}^p \alpha_{\sigma(j)} + 1} \\
&\quad [x_{\sigma(1)}, \dots, N^{\alpha_r} x_{\sigma(s)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \Big) \\
&= \sum_{p=1}^n \left(\sum_{\sigma, \sigma(s)=\alpha_r} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1, j \neq s}^p \alpha_{\sigma(j)} + \alpha_r + 1} \right. \\
&\quad [x_{\sigma(1)}, \dots, x_{\sigma(s)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad + \sum_{\sigma, \sigma(s)=\alpha_r} (-1)^{\frac{(p-1)(p-2)}{2} + \sum_{j=1, j \neq s}^p \sigma(j)} N^{\sum_{j=1, j \neq s}^p \alpha_{\sigma(j)} + 1} \\
&\quad [x_{\sigma(1)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_r} x_{\sigma(s)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \Big) \\
&= N \left(\sum_{p=1}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \beta_{\sigma(j)}} \right. \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\beta_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\beta_{\sigma(n)}} x_{\sigma(n)}] \\
&\quad \left. + [Nx_1, Nx_2, \dots, N^{\beta_r} x_r, N^{\beta_{r+1}} y_{r+1}, \dots, N^{\beta_{n-1}} y_{n-1}, N^{\beta_n} y_n] \right),
\end{aligned}$$

where $\beta_i = 1, 1 \leq i \leq r-1$ and $\beta_i = \alpha_i, r \leq i \leq n$. Therefore, Eq. (24) holds for $\alpha_1 = 1, \alpha_2 = 1, \dots, \alpha_{r-1} = 1$ and arbitrary positive integer $\alpha_r, \alpha_{r+1}, \dots, \alpha_{n-1}, \alpha_n$. In particular, Eq. (24) holds for arbitrary positive $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$.

Suppose that N is invertible. Applying N^{α_n} to Eq. (24), substituting $x'_n = N^{\alpha_n} x_n$, we get

$$\begin{aligned}
&\sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)} - \alpha_n} \\
&\quad [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
&= \sum_{p=0}^n \left(\sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \alpha_{\sigma(j)} - \alpha_n} \right. \\
&\quad \left. [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, x'_n] \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j) + n} N^{\sum_{j=1}^{p-1} \alpha_{\sigma(j)}} \\
& [x_{\sigma(1)}, x_{\sigma(2)}, \dots, N^{-\alpha_n} x'_n, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
& = - \sum_{p=0}^n \left(\sum_{\sigma} (-1)^{\frac{p(p+1)}{2} + \sum_{j=1}^p \sigma(j) + n} N^{\sum_{j=1}^p \alpha_{\sigma(j)} - \alpha_n} \right. \\
& [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, x'_n, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\alpha_{\sigma(n)}} x_{\sigma(n)}] \\
& + \sum_{\sigma} (-1)^{\frac{(p-2)(p-1)}{2} + \sum_{j=1}^{p-1} \sigma(j)} N^{\sum_{j=1}^{p-1} \alpha_{\sigma(j)}} \\
& [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p-1)}, N^{\alpha_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{-\alpha_n} x'_n] \Big) \\
& = \sum_{p=0}^n \sum_{\sigma} (-1)^{\frac{p(p-1)}{2} + \sum_{j=1}^p \sigma(j)} N^{\sum_{j=1}^p \beta_{\sigma(j)}} \\
& [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(p)}, N^{\beta_{\sigma(p+1)}} x_{\sigma(p+1)}, \dots, N^{\beta_{\sigma(n)}} x_{\sigma(n)}] = 0,
\end{aligned}$$

where $\beta_i = \alpha_i$, $1 \leq i \leq n-1$ and $\beta_n = -\alpha_n$. Then the formula (24) holds for $\alpha_i > 0$, $1 \leq i \leq n-1$ and $\alpha_n < 0$. Similarly, the formula (24) holds for $\alpha_i > 0$, $1 \leq i \leq n$, $i \neq j$ and $\alpha_j < 0$. To prove Eq. (24), for $\alpha_{i_1} < 0, \alpha_{i_2} < 0, \dots, \alpha_{i_r} < 0$ and others positive, apply $N^{-\sum_{j=1}^r \alpha_{i_j}}$ to Eq. (24) putting $x'_{i_j} = N^{\alpha_{i_j}} x_{i_j}$, $1 \leq j \leq r$. This ends the proof. ■

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